ON THE POSSIBILITY OF A CONNECTION BETWEEN THE CONSTRUCTION OF A CLASS OF BIGEODETIC BLOCKS AND THE EXISTENCE PROBLEM FOR BIPLANES

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Graph theory and enumerative combinatorics are two branches of mathematical sciences that have developed astonishingly over the past one hundred years. It is especially important to point out that graph theory employs combinatorial techniques to solve key problems of characterization, construction, and enumeration of an enormous set of different classes of graphs. This manuscript describes the construction of two classes of bigeodetic blocks using balanced incomplete block designs (BIBDs). On the other hand, even though graph theory and combinatorics have a close relationship, the opposite problem; that is, considering certain graph constructions when solving problems of combinatorics, is not common, but perfectly possible. The construction of the second class of bigeodetic blocks described in this manuscript represents an example of how graph theory could somehow give a clue to the description of a problem of existence in combinatorics. We refer to the problem of existence for biplanes. The content development of this research also suggests the possibility of a connection between the mentioned construction, the Bruck-Ryser-Chowla theorem, and the problem of existence for biplanes.

Keywords: Bigeodetic Blocks, BIBDs, Biplanes.

1 Introduction

A balanced incomplete block design (or simply a block design) on a set S with $|S| = n$, is a family of subsets $B_1, B_2, \ldots, B_b$ of S called blocks such that:

(a) $|B_i| = k$, $1 \leq i \leq b$.

(b) If $x \in S$, then x belongs to exactly r blocks B_i.

(c) If x, y are distinct elements of S, then $\{x, y\}$ is contained in exactly λ blocks.

This block design is denoted by (b, n, r, k, λ).

Just like any other combinatorial structure, block designs are defined in terms of certain parameters whose values determine the answer to the question of existence; that is, which values of these parameters produce the configuration in question and which do not. Given a (b, n, r, k, λ)-design, there are necessary conditions that its parameters must satisfy, namely, $bk = nr$ and $r(k - 1) = \lambda(n - 1)$.

A design with $b = n$ is called symmetric. In such a design $r = k$ and hence such structure is called (n, k, λ)-design. For symmetric designs, there is an additional restriction for their existence [4, Theorem 3.1].

Date: February 20, 2010

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Theorem 1. (Bruck-Ryser-Chowla Theorem) Let n, k, λ be integers for which there exists a symmetric (n, k, λ)-design. If n is even, then $k - \lambda$ equals a perfect square. If n is odd, then the Diophantine equation

$$x^2 = (k - \lambda)y^2 + (-1)^{(n-1)/2}\lambda z^2$$

has a solution in integers x, y, z not all zero.

The conditions previously described on the parameters of a block design are necessary, but not sufficient. It means that we can use them to rule out the existence of a block design for certain groups of parameters. However, being given the values of the parameters which satisfy the conditions previously mentioned does not guarantee the existence of a block design with those parameters. There are many groups of possible parameters for which the existence problem has not been settled.

In this research a graph is undirected, without loops or multiple edges. Bigeodetic graphs were defined by Srinivasan [5, p.102] as graphs in which each pair of nonadjacent vertices has at most two paths of minimum length between them. K-geodetic graphs have been defined in [1, p. 188] as graphs in which each pair of nonadjacent vertices has at most k paths of minimum length between them. Thus, a k-geodetic graph is geodetic when $k = 1$, bigeodetic when $k = 2$, trigeodetic when $k = 3$, and so on. A block is a graph with vertex connectivity > 1. In [1, pp. 190-201], a general study of k-geodetic graphs has been performed and bigeodetic blocks have been considered there as a particular case of k-geodetic graphs.

A *cover* of a graph G is a set $\{G_1, G_2,...,G_m\}$ of complete subgraphs of G such that $G_1 \cup G_2 \cup ... \cup G_m = G$. A cover of G is called a Θ-*cover* if any two elements of the cover are edge-disjoint.

Let G be a graph having vertices $v_1, v_2,...,v_n$. Let $A = \{G_1, G_2,..., G_m\}$ be a cover of G, where $V(G_i) = \left\{v_{i_1}, v_{i_2},...,v_{j_i}\right\}$, $1 \le i \le m$. For each i, $1 \le i \le m$, take new vertices $v_{i_1 i}, v_{i_2 i},...,v_{i_{j_i} i}$ and construct a complete graph $K(G_i)$ on these vertices. Take n new vertices $v_{10}, v_{20},...,v_{n0}$ and connect $v_{i_\ell i}$ to $v_{i_\ell 0}$ for $1 \le \ell \le j_i$, $1 \le i \le m$. The resulting graph is denoted $G^*(A)$.

Consider a (b, n, r, k, λ)-design on a set $S = \{x_1, x_2,...,x_n\}$. Let K_n be a complete graph with vertex set $\{x_1, x_2,...,x_n\}$ and G_i be a complete graph on vertex set of B_i, $1 \le i \le b$. Clearly, $A = \{G_1, G_2,...,G_b\}$ is a cover of K_n. Construct graph $K_n^*(A)$ and denote it as $K_n^*(r, k, \lambda)$. This is a k-connected, biregular block with degree sequence (r, k). It has $n(r + 1)$ vertices and $nr(k + 1)/2$ edges. In Figure 2, $K_7^*(6,3,2)$ is constructed using the blocks of a (14,7,6,3,2)-design.

The described procedure to generate graph $K_n^*(r, k, \lambda)$ and the following theorem with its respective corollary are taken from [5, pp. 103-107].

Theorem 2. Let $\mu = \max[\max(|B_i \cap B_j| : i, j = 1,...,b, i \ne j), \lambda]$. Any pair of nonadjacent vertices of $K_n^*(r, k, \lambda)$ has at most μ distinct paths of minimum length between them. The diameter of $K_n^*(r, k, \lambda)$ is 4 or 5 according as $B_i \cap B_j \ne \emptyset$ for every i, j or not.

Corollary 1. If (b, n, r, k, λ) is a symmetric design, then in $K_n^*(r, k, \lambda)$ there are at most λ paths of minimum length between each pair of vertices.

2 Results

Next we present two constructions of bigeodetic blocks using block designs (Theorem 3 and Assertion 1).

The construction described in Assertion 1 has a special connotation because even though it describes a simple observation about the existence behavior pattern of the employed symmetric $((n^2+n+2)/2,\ n+1,\ 2)$-designs in a very short interval of integer values, section 3 of this manuscript suggests that this simple observation could not be just a coincidence and could give a clue to the description of a more general problem of existence.

Theorem 3. For every $n \equiv 0$ or 1 (mod 3), $n \geq 4$, there exists a bigeodetic block on n^2 vertices with diameter 4 or 5, with vertex connectivity 3 and degree sequence $(n$-1, 3).

Proof: When $n \equiv 0$ or 1 (mod 3), $n \geq 4$, there exists an $(n(n$-1)/3, n, n-1, 3, 2)-design on a set S [2, Theorem 15.4.5]. Thus, taking G_i to be a complete graph on vertices of B_i, $1 \leq i \leq n(n-1)/3$, graphs $G_1, \ldots, G_{n(n-1)/3}$ form a cover of the complete graph K_n on vertex set S. Construct graph $K_n^*(n$-1, 3, 2). This graph has n^2 vertices. By Theorem 2 this is a bigeodetic graph of diameter 4 or 5 according as $B_i \cap B_j \neq \phi$ for every i, j, $i \neq j$ or not. It is easy to observe that $K_n^*(n$-1, 3, 2) has degree sequence $(n$-1, 3) and is 3-connected for $n \geq 4$.

Next we give the blocks of (10, 6, 5, 3, 2) and (14, 7, 6, 3, 2) designs which are used to construct the bigeodetic blocks shown in Figures 1 and 2.

(i) $\{x_1, x_2, x_4\}$, $\{x_1, x_2, x_3\}$, $\{x_3, x_4, x_5\}$, $\{x_2, x_4, x_5\}$, $\{x_2, x_5, x_6\}$, $\{x_1, x_5, x_6\}$, $\{x_2, x_3, x_6\}$, $\{x_1, x_3, x_5\}$, $\{x_1, x_4, x_6\}$, $\{x_3, x_4, x_6\}$.

(ii) $\{x_1, x_2, x_4\}$, $\{x_1, x_2, x_3\}$, $\{x_3, x_4, x_6\}$, $\{x_3, x_4, x_5\}$, $\{x_2, x_5, x_6\}$, $\{x_3, x_6, x_7\}$, $\{x_1, x_6, x_7\}$, $\{x_1, x_4, x_7\}$, $\{x_2, x_3, x_7\}$, $\{x_1, x_3, x_5\}$, $\{x_2, x_5, x_7\}$, $\{x_2, x_4, x_6\}$, $\{x_1, x_5, x_6\}$, $\{x_4, x_5, x_7\}$.

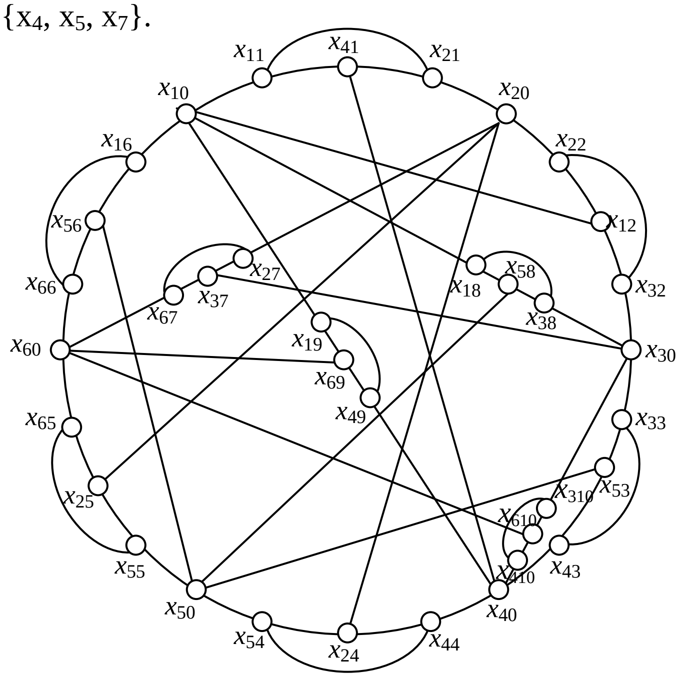

Fig. 1. A bigeodetic block generated by a (10, 6, 5, 3, 2)-design.

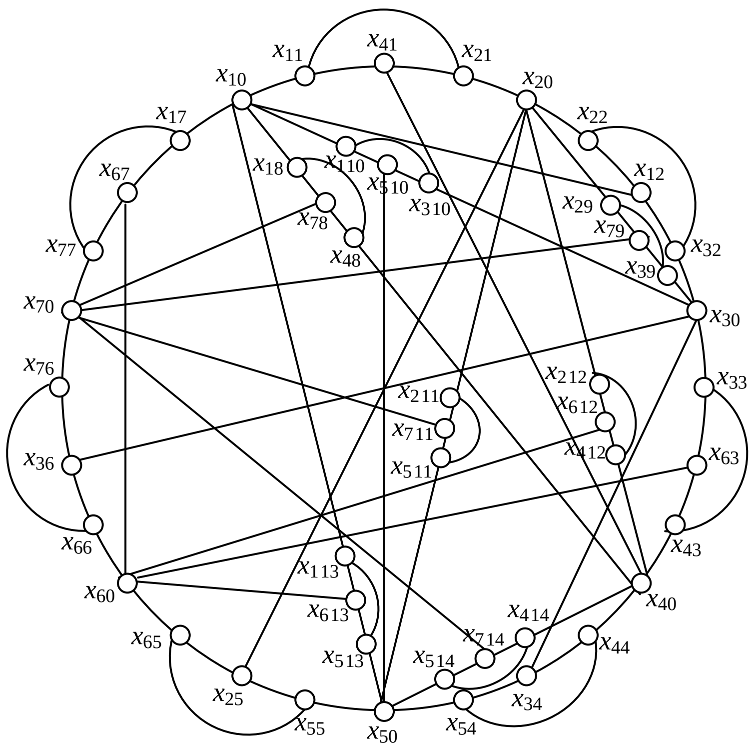

Fig. 2. A bigeodetic block generated by a (14, 7, 6, 3, 2)-design.

Assertion 1. For every $n \equiv 1$ or 2 (mod 4), $2 \le n \le 10$, such that $(n\text{-}1)$ is a perfect square or $n \equiv 0$ or 3 (mod 4), $3 \le n \le 12$, such that $(n\text{-}1)$ is a prime power, there exists an $(n+1)$-regular, $(n+1)$-connected bigeodetic block of diameter 4.

Proof: When $n \equiv 1$ or 2 (mod 4), $2 \le n \le 10$, such that $(n\text{-}1)$ is a perfect square or $n \equiv 0$ or 3 (mod 4), $3 \le n \le 12$, such that $(n\text{-}1)$ is a prime power, there exists a symmetric block design $((n^2+n+2)/2, n+1, 2)$ on a set S with blocks B_i, $1 \le i \le (n^2+n+2)/2$ (Section 3 of this manuscript lists all symmetric $((n^2+n+2)/2, n+1, 2)$-designs so far found. Note that they obey the "simple pattern" of existence mentioned at the beginning of this proof). Let G be a complete graph on vertex set S, and G_i be a complete graph on vertex set B_i, $1 \le i \le (n^2+n+2)/2$. $G_1,...,G_{(n^2+n+2)/2}$ form a cover of G. Construct graph $G^*_{(n^2+n+2)/2}(n+1, n+1, 2)$. This graph is an $(n+1)$-regular, $(n+1)$-connected one and has $(n^2+n+2)(n+2)/2$ vertices. By Corollary 1 this is a bigeodetic block. Since any two blocks of a design $((n^2+n+2)/2, n+1, 2)$ have two common elements, the diameter of $G^*_{(n^2+n+2)/2}(n+1, n+1, 2)$ is 4.

Next we give the blocks of (4, 4, 3, 3, 2) and (7, 7, 4, 4, 2) designs which are used to construct the bigeodetic blocks shown in Figures 3 and 4.

(i) $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_1, x_3, x_4\}$, $\{x_2, x_3, x_4\}$.

(ii) $\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_3, x_5, x_7\}$, $\{x_1, x_4, x_5, x_6\}$, $\{x_1, x_2, x_6, x_7\}$, $\{x_2, x_3, x_5, x_6\}$, $\{x_2, x_4, x_5, x_7\}$, $\{x_3, x_4, x_6, x_7\}$.

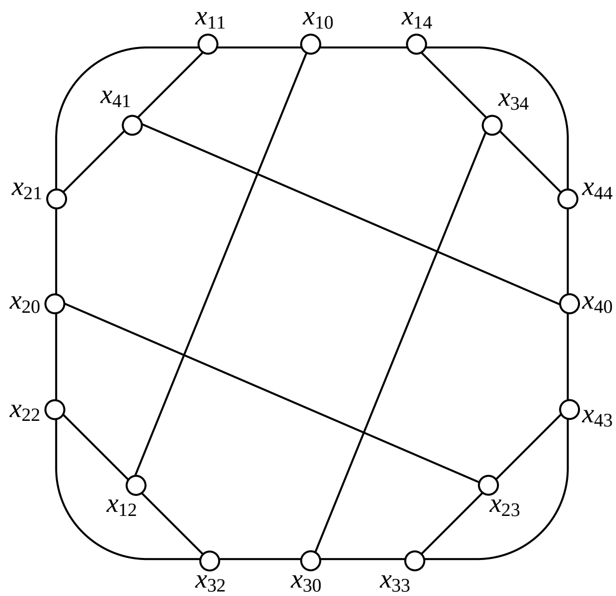

Fig. 3. A bigeodetic block generated by a (4, 4, 3, 3, 2)-design.

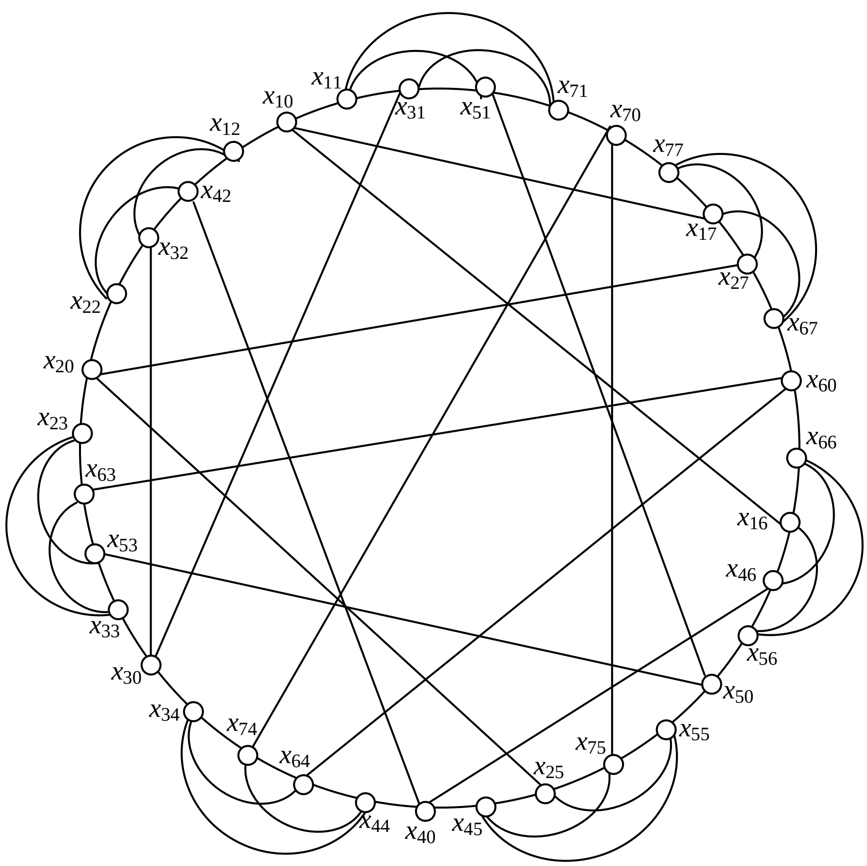

Fig. 4. A bigeodetic block generated by a (7, 7, 4, 4, 2)-design.

Corollary 1. If (b, n, r, k, 2) is a block design, then K_n*(r, k, 2) is a bigeodetic graph of diameter either 4 or 5.

Srinivasan, Opatrny, and Alagar [5, p. 111] considered that it is possible to construct n-regular, k-connected bigeodetic blocks of diameter d, where $k, n, d \geq 2$, $k \leq n$.

They have denoted this class of blocks as $B(k, n, d)$ and have posed the following problem:

Is class $B(k, n, d)$ nonempty for every $k, n, d \geq 2$, $k \leq n$?

Assertion 2. For every $n \equiv 1$ or 2 (mod 4), $2 \leq n \leq 10$, such that (n-1) is a perfect square or $n \equiv 0$ or 3 (mod 4), $3 \leq n \leq 12$, such that (n-1) is a prime power, class $B(n+1, n+1, 4)$ is nonempty.

3 Conclusions

Similar constructions to those ones described in Theorem 3 and Assertion 1 for bigeodetic blocks can be formulated for trigeodetic blocks using (b, n, r, k, 3)-designs. Thus, when $n \equiv 1$ (mod 2), $n \geq 5$, there exists an ($n(n$-1)/2, n, 3(n-1)/2, 3, 3)-design [2, Theorem 15.4.2]. In the same way, when $n \equiv 0$ or 2 (mod 3), $3 \leq n \leq 14$, $n \neq 12$, there exists a symmetric ((n^2+n+3)/3, n+1, 3)-design, namely, (5, 4, 3), (11, 6, 3), (15, 7, 3), (25, 9, 3), (31, 10, 3) [2, Appendix 1], (45, 12, 3), (71, 15, 3) [3, p. 105], which is called a *triplane*.

For any fixed integer value $\lambda \geq 2$, the question of whether there exists an infinite number of symmetric (n, k, λ)-designs is unresolved. In particular, when $\lambda = 2$, such a design is called a *biplane* and there exists only a few known examples, namely, (4,3,2), (7,4,2), (11,5,2), (16,6,2), (37,9,2), (56,11,2), and (79,13,2). The first two biplanes are here used to generate two bigeodetic blocks (see Figures 3 and 4). Ryser [4, pp.114-115] proved that if in a symmetric (n, k, λ)-design n is odd and (k, λ) = 1 where (k, λ) denotes the positive greatest common divisor of k and λ, then (k - λ, λ) = 1 and the Diophantine equation $x^2 = (k - \lambda)y^2 + (-1)^{(n-1)/2}\lambda z^2$ associated to the Bruck-Ryser-Chowla theorem has a solution in integers x, y, and z, not all zero. It is evident that for $n \equiv 0$ or 3 (mod 4) with $n > 3$ and (n-1) a prime power, (n^2+n+2)/2 is odd and ((n^2+n+2)/2, n+1, 2)-biplanes satisfy the conditions established by Ryser. As a result, when changing n, k, and λ in $x^2 = (k - \lambda)y^2 + (-1)^{(n-1)/2}\lambda z^2$ by (n^2+n+2)/2, n+1, and 2, respectively, a Diophantine equation with a solution in integers x, y, and z, not all zero is generated. Consequently, for $n \equiv 0$ or 3 (mod 4) with $n > 3$ and (n-1) a prime power, (n^2+n+2)/2 is odd and ((n^2+n+2)/2, n+1, 2)-biplanes satisfy the necessary condition established in Theorem 1 for their existence.

It has been conjectured that only finitely many symmetric designs exist for any fixed $\lambda > 1$. Assuming that this is true, one could speculate if for a given finite integer interval, biplanes and their existence respond to the same simple pattern of behavior described in the construction of Assertion 1's bigeodetic blocks.

Assume that n belongs to a finite interval of integer values [2, m], m a fixed integer, $m \geq 12$. Could it be possible that being $n \equiv 1$ or 2 (mod 4), $2 \leq n < m$, such that (n-1) is a *perfect square or* $n \equiv 0$ *or 3 (mod 4),* $3 \leq n \leq m$*, such that (n-1) is a prime power, there exists a symmetric block design* $((n^2+n+2)/2, n+1, 2)$*?*

Note that the answer to this question is affirmative for $m = 12$.

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