Graph Theory and the Problem of Coloring Octahedrons with Six Colors as a Generalization of the Four-Colored Cube Game

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Dedicated to my Mentor and Great Friend Carlos Zuluaga

Introduction

The game of the four colored cubes deals with four cubes having faces colored arbitrarily with four colors, such that each color appears on at least one face of each cube. The object of the game is to stack the cubes in a vertical rectangular prism so that, if possible, all four colors appear on each long face of the prism. The game has been in existence since at least the 1940s, when it was sold commercially under the name “Tantalizer”, and more recently as the popular puzzle “Instant Insanity”. The game that I am trying to describe consists of six octahedrons having faces colored arbitrarily with six colors, such that each color appears on at least one face of each octahedron. The object of this game is similar. You have to stack the octahedrons in a vertical geometric solid so that, if possible, all six colors appear on each long side of the obtained solid. Note that the long sides of the obtained geometric solid are not completely flat.

Description of the Game: Problem and Solution

We will exhibit each octahedron unfolded. If the six colors are blue (B), green (G), orange (O), red (R), violet (V), and yellow (Y), six octahedrons arranged as in Figure 1 will provide a solution to the puzzle.

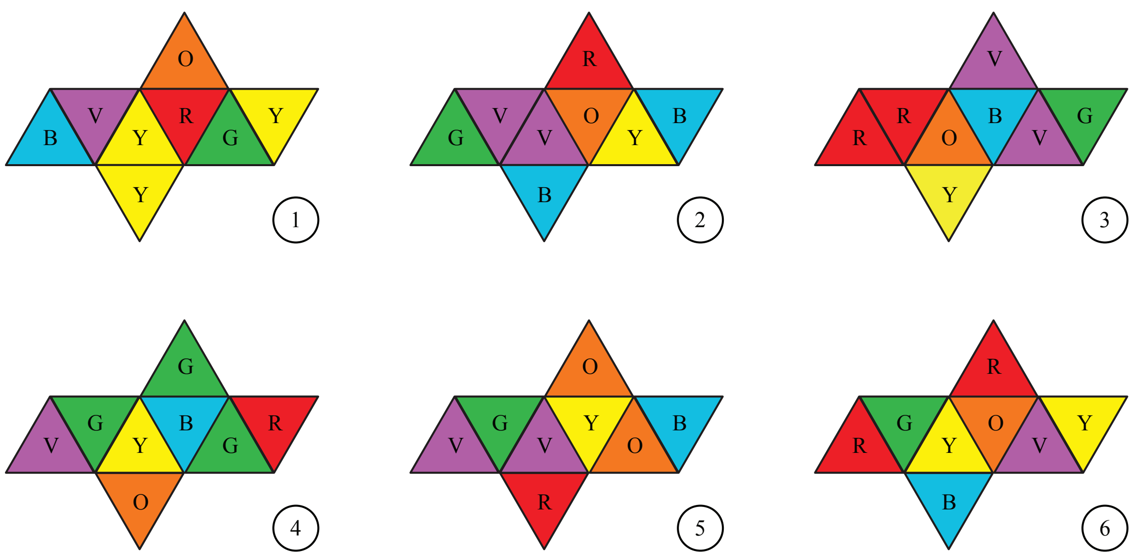

Figure 1. Octahedrons 1, 2, 3, 4, 5, 6.

In order to understand why we need a method of simplifying the problem, let’s examine in how many different ways these six octahedrons can be arranged.

There are only four ways to arrange the first octahedron. They are determined by its four pairs of opposite faces. In the case of our geometric solid, the second octahedron can be rotated and placed in three different ways for each of its eight faces. This determines a total of $3 \cdot 8 = 24$ arrangements. The remaining octahedrons can be also arranged in 24 different ways. Thus, the total number of arrangements of the six octahedrons is $4 \cdot 24 \cdot 24 \cdot 24 \cdot 24 \cdot 24 = 31{,}850{,}496$. So, picking an arrangement which is a solution could be extremely difficult.

In the case of the disposition of colors shown in Figure 1, my goal is to use graph theory to simplify the search for a solution to the puzzle.

Define a graph **Γ** with six vertices labeled B, G, O, R, V, and Y. If octahedron *i* has a pair of opposite faces with colors *x* and *y*, draw an edge with label *i* between vertices *x* and *y*.

The graph of Figure 2 is the graph associated with the octahedrons of Figure 1. Note that both loops and multiple edges are allowed in this labeled graph. Note also that the degree of each vertex of the graph of Figure 2 represents the total number of faces with the corresponding color.

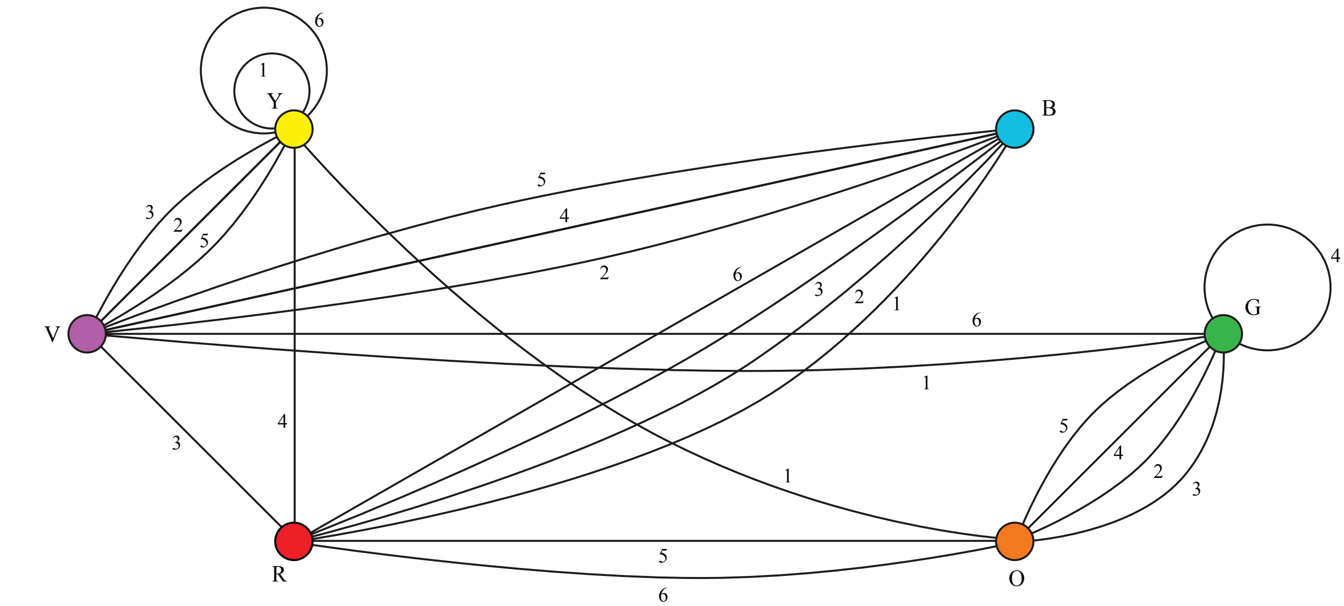

Figure 2. Graph associated with the octahedrons of Figure 1.

The solid that we want to construct has three pair of opposite long sides. Let's name them *first*, *second*, and *third* pairs of opposite long sides. A subgraph of **Γ** with six edges will determine the twelve faces of the first pair of long sides and each of its six edges will have labels 1, 2, 3, 4, 5, and 6. No color will appear twice on the first, second, or third pair of opposite long sides if and only if each vertex of the subgraph has degree two. Employing the same principle for the second and third pair of opposite long sides, we obtain the subgraphs of Figure 3. Note that these three subgraphs are edge-disjoint that is to say; they don't possess edges labeled with the same number joining exactly the same pair of vertices. So, in order to obtain a solution to the puzzle, these three subgraphs must satisfy the conditions established in the following generalization.

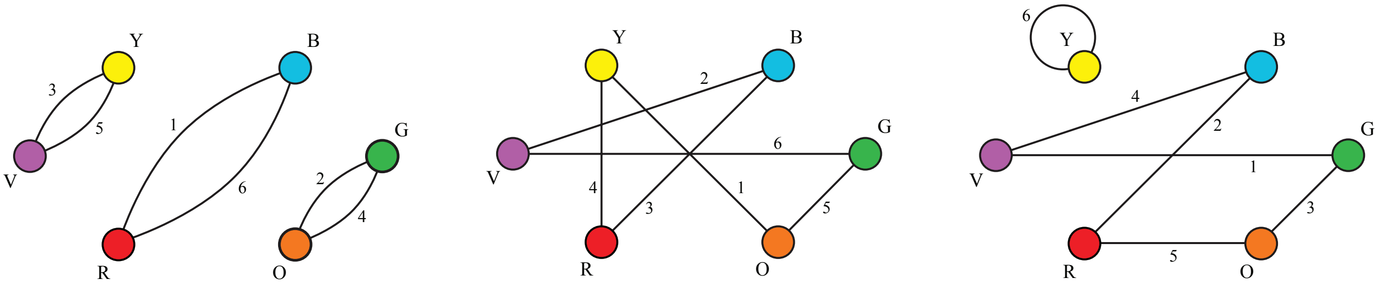

Figure 3. Subgraphs for the first, second, and third pair of opposite long sides from left to right.

Theorem 1. A set of colored octahedrons has a solution if and only if its corresponding labeled graph has three edge-disjoint subgraphs, each of which has every vertex of degree two and every edge label 1 to 6 appearing exactly once.

Figure 4 exhibits the obtained geometric solid seen from four different angles. The two sketches to the left represent the solid seen from front and back. The two to the right are lateral sides of the geometric solid.

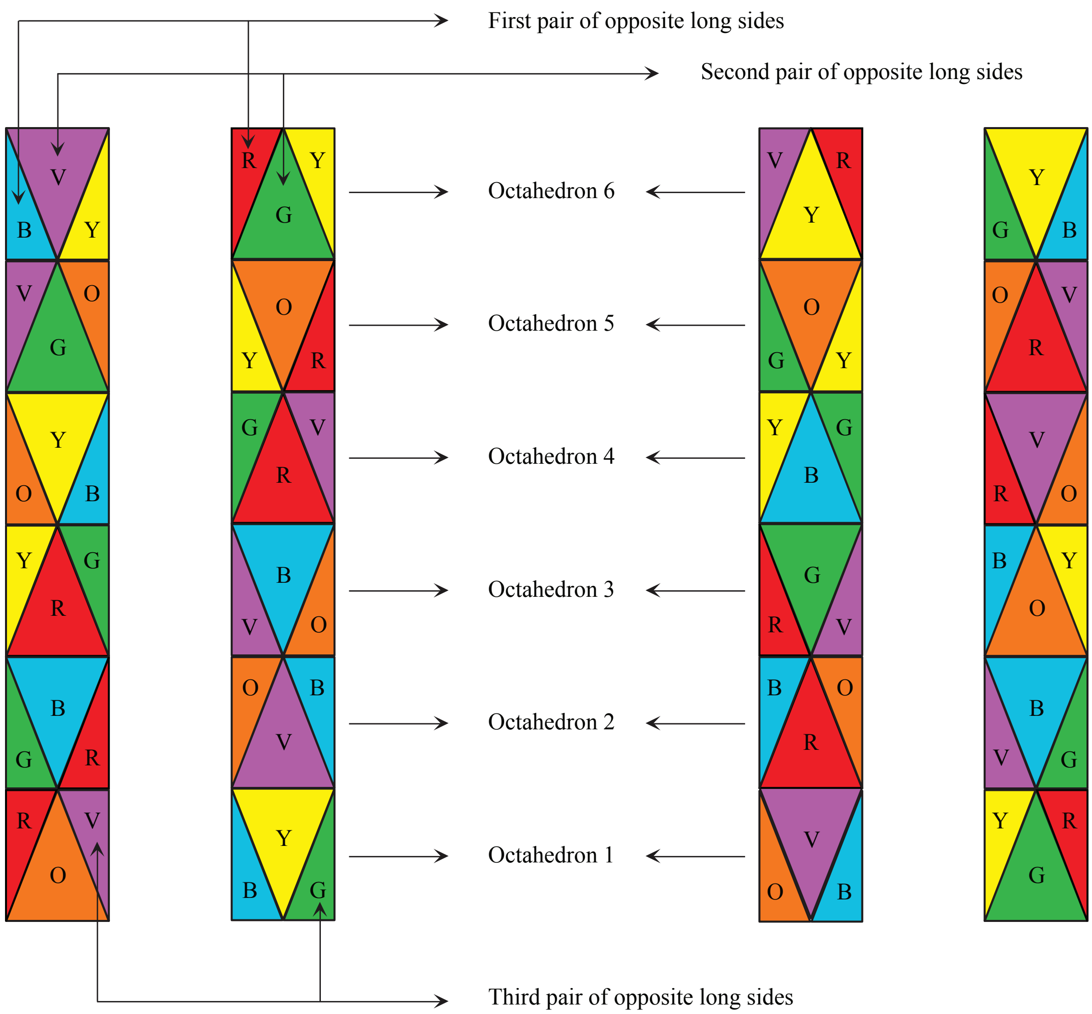

Figure 4. The geometric solid seen from four different angles.

Problem

Write a program that takes as input the pattern of colors on a set of six colored octahedrons and finds all solutions (if there are any).

References

R. Wilson, *Introduction to Graph Theory*, Addison Wesley, 4th Edition, London, 1996.